Math 10A
Worksheet, Discussion #9; Thursday, 6/28/2018
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1 Optimization

1. Suppose you are trying to make a rectangular fence for your yard. You only have $100m$ of fence but luckily your house borders a straight river, so one side of your rectangular yard will be bordered by a river. What is the largest area yard you can enclose?

   Solution: Let s be the side length of the yard that is perpendicular to the river. Then the side length of the yard that is parallel to the river is $(100 - 2s)$ and the area of the yard is $A(s) = s(100 - 2s)$. Taking the derivative gives $A'(s) = 100 - 4s$. So $A' = 0$ when $s = 25$ and note that $A''(s) = -4 < 0$ so this means that $A(25)$ is a local maximum. The largest area is $A(25) = 25(50) = 1250m^2$.

2. You want to construct a cylindrical container that contains $100\pi m^3$ of water. What should the dimensions of the container be if you want to minimize the total surface area?

   Solution: The surface area is $S(r, h) = 2(\pi r^2) + 2\pi rh$. The volume of the container is $V = 100\pi = \pi r^2 h$. So $h = \frac{100}{r^2}$ and so $S(r) = 2\pi r^2 + \frac{200\pi}{r}$. Taking the derivative and setting it equal to 0 gives $4\pi r - \frac{200\pi}{r^2} = 0$ so $r^3 = 50$ so $r = \sqrt[3]{50}$.

3. An airline is selling tickets for \$200 each and sells 50 per plane. For every \$10 they decrease the price, they sell 10 more tickets. The plane can hold a maximum of 100 passengers. At what price should they sell their tickets for maximum revenue?

   Solution: Let x be the amount they decrease the price. Then at a price of $200 - x$ each, they sell $50+x$ tickets. So the total revenue is $R(x) = (200-x)(50+x)$. Taking the derivative, we get $R'(x) = 150 - 2x$. Setting the derivative to 0, we get that $x = 75$ so we should sell $50 + 75 = 125$ tickets. But since the plane has a maximum of 100 passengers and $R'(x)$ for all $50 + x < 125$, this tells us that $x = 50$ is the maximum on the domain of x which is $\{x : x \leq 50\}$. So they should set a price of $200 - 50 = \$150$.

4. Find the rectangle of largest area whose diagonal is of length L.

Solution: Let one of the side lengths of the rectangle by s, then finding the largest area is the same as finding the largest area squared which is $s^2(L^2 - s^2)$. Taking the derivative and setting it equal to 0 gives $2L^2 s - 4s^3 = 0$ so $s = 0$ or $s = \frac{L}{\sqrt{2}}$. At $s = 0$, the second derivative is positive and at $s = L/\sqrt{2}$, the second derivative is negative which tells us that $s = L/\sqrt{2}$ gives us the largest area. The other side length is $\sqrt{L^2 - L^2/2} = L/\sqrt{2} = s$ so the largest area is achieved with a square.

5. Find the area of the smallest triangle formed by the x axis, y axis, and a line that goes through the point $(4, 2)$.

Solution: Suppose that the line goes through the point $(0, y_0)$. Then, the slope of the line is $\frac{2-y_0}{4}$ and is described by the line $y - y_0 = \frac{2-y_0}{4}x$. The x intercept is when $y = 0$ or when $x = \frac{4y_0}{y_0-2}$. Thus, the area of the triangle is

$$A(y_0) = \frac{1}{2} \cdot y_0 \cdot \frac{4y_0}{y_0 - 2} = \frac{2y_0^2}{y_0 - 2}.$$

Setting the derivative equal to zero gives $A'(y) = \frac{2y(y-4)}{(y-2)^2}$ so the two solutions are $y = 0$ and $y = 4$. The second derivative is $\frac{16}{(y-2)^3}$ and so $y_0 = 0$ is a local maximum and $y_0 = 4$ is a local maximum. So the area is $\frac{2 \cdot 4^2}{4-2} = 16$.

6. Find the largest rectangle that can be inscribed into a semicircle of radius 1 so that one side of the rectangle is part of the diameter of the semicircle.

Solution: Let the height of the rectangle be h. Then the other side of the rectangle must be $2\sqrt{1-h^2}$. So we want to maximize $2h\sqrt{1-h^2}$, which is the same as maximizing its square $4h^2(1-h^2)$. Setting the derivative equal to 0 gives $8h - 16h^3 = 0$ so $h = 1/\sqrt{2}$. The area is $2/\sqrt{2} \cdot 1/\sqrt{2} = 1$.

7. Suppose you only have $1m$ of wire. You are to construct a circle and a square. What is the maximum and minimum total area of the circle and square?

Solution: Let s be the side length of the square and r be the radius of the circle. Then $4s + 2\pi r = 1$ so $r = \frac{1-4s}{2\pi}$. So the total area is

$$A(s) = s^2 + \frac{\pi(1-4s)^2}{4\pi^2}.$$

Setting the derivative equal to 0 gives $s = \frac{1}{\pi+4}$ and the second derivative is $2 + \frac{8}{\pi}$ which is always positive. Thus, $s = \frac{1}{\pi+4}$ is a local minimum and $A = \frac{1}{16+4\pi}$ is the minimum area. The domain of s is $[0, 1/4]$ so the other critical points are the end point. We have that $A(0) = 1/4\pi$ and $A(1/4) = 1/16$ so the maximum area is $1/4\pi$ which occurs at $s = 0$ so we only make a circle.